LECTURE 9

In the last lecture, we ended on the classical form of the derivative, i.e.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

But the philosophy of having $h \to 0$ is equivalent to have a point $z \to x$. Therefore, an alternate form of the derivative is

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}$$

(essentially with a change of variable $z = x + h$).

Example 1. (Using the alternate definition) Find the derivative of $f(x) = \sqrt{x}$.

Solution. Consider the alternate definition,

$$\begin{aligned} f'(x) &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\ &= \lim_{z\to x} \frac{1}{\sqrt{z} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

which is cleaner than using conjugation in the classical form.

Graphing the derivative $f'(x)$ informs you about how fast the original graph of $f(x)$ is changing. Consider example 1 (also a brilliant example in Figure 3.6 of the book).

Definition 2. (Left and Right Derivative)

$$\lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h}, \quad \text{left-hand derivative at } x = a$$

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}, \quad \text{right-hand derivative at } x = a$$

Example 3. Find the derivative of $f(x) = |x|$ and check if derivative exists at $x = 0$.

Solution. Note that separately for $x < 0$ and $x > 0$, the graph looks like two lines which are certainly differentiable. The point of controversy is $x = 0$. Left-hand derivative at $x = 0$

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

while right-hand derivative at $x = 0$

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Since the left-hand and the right-hand derivatives are NOT equal, the function is NOT differentiable at $x = 0$. Also, see the graph for an intuition (you can put infinitely many tangent lines at $x = 0$, which means infinitely many slopes, but the derivative is unique).

Theorem 4. *(Differentiability implies continuity) If f has a derivative at $x = c$, then f is continuous at $x = c$.*

Proof. So, we know $f'(c)$ exists (by the premise). We **want** to show $\lim_{x\to c} f(x) = f(c)$ which is the definition of continuity. An equivalent statement would be $\lim_{h\to 0} f(c+h) = f(c)$ (you are approaching c closer and closer).

Your goal is to use the information that $f'(c)$ is something concrete while evaluating $\lim_{h\to 0} f(c+h)$ (and see if it is equal to $f(c)$). Thus, you are trying to "cook" up a situation where $f'(c)$ shows up somehow when evaluating $\lim_{h\to 0} f(c+h)$, by means of proper mathematical operations. We do the infamous, **add and subtract**, and then **multiply and divide:**

$$\begin{aligned}\lim_{h\to 0} f(c+h) &= \lim_{h\to 0} f(c+h) - f(c) + f(c)\\ &= f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} h\\ &= f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \lim_{h\to 0} h\\ &= f(c) + f'(c)\cdot 0\\ &= f(c)\end{aligned}$$

Using the alternate definition of the derivative, we can also do this (we start with $\lim_{x\to c} f(x)$)

$$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c} f(x) - f(c) + f(c)\\ &= f(c) + \lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x-c)\\ &= f(c) + \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \lim_{x\to c}(x-c)\\ &= f(c) + f'(c)\cdot 0\\ &= f(c)\end{aligned}$$

□

Remark. If f is continuous at $x = c$, it does NOT imply that f has a derivative at $x = c$.

Counterexample: $f(x) = |x|$ at $x = 0$. Continuous but not differentiable at $x = 0$.

Thus, we say, it is sufficient that f is differentiable for it to be continuous. (Differentiable is a higher order of smoothness).

Do exercise 45-50.